A differential structure for quantum mechanics

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Abstract. *A method of obtaining a non-commutative analogue of a differential structure from the action of a Lie group on a C^*-algebra is proposed. The addition of this structure to the usual structure of quantum mechanics turns out to be equivalent to the replacement of the Hilbert space by a Gelfand triple (rigged Hilbert space).*

INTRODUCTION

It is striking that the notion of a «smooth classical observable» (smooth function on the phase space) has no quantum analogue up to now. This paper attempts to fill this gap. The idea is simple: smooth functions can be characterized as those functions which depend smoothly on translations of the underlying space. We can apply this characterization to the case of quantum-mechanical observables, since there is a natural group of translations in quantum mechanics (Sec. 3). In this paper we consider quantum-mechanical systems corresponding to flat configuration spaces.

From the mathematical point of view, we try to extend the category of differential manifolds to include non-commutative objects. It is known that the category of locally compact spaces has a natural extension given by the theory of C^*-algebras («pseudospaces» of [1], see also [2]). One expects that the analogue of a differential structure on a locally compact space is a particular dense $*$-sub-

Key-Words: *Non-commutative differential structures - C^* algebras.*
1980 Mathematics Subject Classification: *19 F 99, 81 C 99.*

algebra A^∞ of a C^*-algebra A (in the case of a differential manifold M, A is composed of continuous functions on M, vanishing at infinity, and A^∞ is composed of smooth functions belonging to A). In the present paper we propose a method of obtaining A^∞ from an action of a Lie group on A. Our method is different from that of [2] and leads to the original differential structure in the commutative case (whereas the method of [2] does not).

The case of quantum mechanics is studied in Sec. 3. We show that the resulting differential structure can be equivalently described by a particular dense subspace in the Hilbert space of quantum theory (this subspace turns out to be the Schwartz space $\mathscr{S}$ in the Schrödinger representation). This gives a connection between the differential structure and the «Gelfand triple» structure. We hope this fact will help us to answer the question how some important symplectic-geometrical constructions can be performed in the case of quantum mechanics. Some results concerning this problem can be found in [3] and [4]. In those papers we faced the problem of «choosing the correct Gelfand triple» and we have chosen the one associated with the space $\mathscr{S}$ just for convenience. The present paper justifies this choice to some extent.

Our analysis of noncommutative differential structures is not complete and some questions remain unanswered. The commutative case suggests answers to these questions. We consider these answers as desirable «consistency conditions» in the non-commutative case. In the case of quantum mechanics we are able to prove all these conditions except one. Whether it is satisfied or not seems to be an interesting mathematical problem.

1. PSEUDOSPACES

In this section we recall the sense in which the theory of C^*-algebras is a generalization of the theory of locally compact spaces (see [1] for details).

There is a bijective correspondence between locally compact spaces Λ and commutative C^*-algebras A, given by the relation $A = C_\infty(\Lambda)$ (complex continuous functions on Λ, vanishing at infinity). If Λ_1 and Λ_2 are two locally compact spaces, then the continuous mappings $f : \Lambda_1 \to \Lambda_2$ are in bijective correspondence with the elements of

(1) $\mathrm{Mor}(A_2, A_1) = \{\varphi : \varphi$ is a $*$-homomorphism of A_2 into $M(A_1)$, such that the set $A_1\,\varphi(A_2)$ is total in $A_1\}$,

where $A_k = C_\infty(\Lambda_k)$, the correspondence between f and φ being given by $\varphi(a_2) = a_2 \circ f$ for $a_2 \in A_2$. Here for any C^*-algebra A we denote by $M(A)$ the multiplier algebra of A ([9], [1]). The algebra $M(A)$ can be defined as follows. One can always embed A into the algebra $B(H)$ of all bounded operators in a Hilbert space H. Let us fix one such embedding $A \subset B(H)$ and put

$$M(A) = \{ b \in B(H) : bA \subset A, Ab \subset A \}.$$

$M(A)$ is a C^*-algebra and contains A as an ideal. The C^*-algebra $M(A)$ is determined only by A and does not depend on the way we embed A into $B(H)$, i.e. different embeddings lead to isomorphic C^*-algebras (see [1]). For $A = C_\infty(\Lambda)$, $M(A)$ is the algebra of bounded continuous functions on Λ.

We conclude that the category of locally compact spaces is dual to the category of commutative C^*-algebras with morphisms defined in (1).

All (not necessarily commutative) C^*-algebras with morphisms defined as in (1) form a category. The possibility of composing morphisms is guaranteed by the fact that any $\varphi \in \mathrm{Mor}\,(A_2, A_1)$ can be uniquely extended to a homomorphism $\tilde{\varphi} : M(A_2) \to M(A_1)$. The objects of the (formally) dual category are called pseudospaces. «Pseudospace» is (in general) only a suggestive name for a C^*-algebra A if one wants to think of A as an analogue of a locally compact space. We have the following analogies:

$\mathrm{Mor}\,(C_\infty(\mathbb{R}), A)$ – «real continuous functions»

$\mathrm{Re}\, M(A)$ – «bounded real continuous functions»

$\mathrm{Re}\, A$ – «real continuous functions, vanishing at infinity»,

where $\mathbb{R}$ is the real line. Here for any $*$-algebra B we denote by $\mathrm{Re}\, B$ the set of self-adjoint elements of B. The inclusion of $\mathrm{Re}\, M(A)$ into $\mathrm{Mor}\,(C_\infty(\mathbb{R}), A)$ is given by assigning to each $a = a^* \in M(A)$ the homomorphism $C_\infty(\mathbb{R}) \ni f \mapsto f(a) \in M(A)$. Here $f(a)$ is defined by the functional calculus of self-adjoint elements in C^*-algebras (see [7]). We shall use also the analogy

$\mathrm{Re}\, A_0$ – «real continuous functions with compact supports»,

where A_0 is the «minimal dense ideal» of Pedersen [5] in A.

2. DIFFERENTIAL STRUCTURE INDUCED BY AN ACTION OF A LIE GROUP - A GENERAL SCHEME

Suppose we are given an action of a Lie group G on a C^*-algebra A. This means, we have a homomorphism

$$G \ni g \longmapsto \sigma_g \in \mathrm{Aut}\, A,$$

such that for every $a \in A$, the mapping $G \ni g \longmapsto \sigma_g a \in A$ is norm-continuous. In this case we call (A, G, σ) a C^*-dynamical system.

If $A = C_\infty(\Lambda)$ then such an action is equivalent to the action of G on Λ. If the latter action is *transitive*, it naturally induces a differential structure on Λ. If the action is not transitive, it still enables us to distinguish functions «differentiable in directions tangent to orbits». This corresponds to a more general notion of a differential structure than the manifold structure. In the sequel we adopt this more general point of view, since it is not quite clear what is the non-commutative analogue of transitivity (the transitivity is related to the nuclearity of algebras of smooth functions – this was pointed to me by the Author of [1]). This point is not essential for the general scheme we want to formulate in this section. The nest section is devoted to an example of (A, G, σ) where the action σ is ergodic. Ergodicity can be considered as a first step to transitivity (apparently, the latter property is stronger than the former). An action σ of G on A is *ergodic*, if the condition

$$\sigma_g a = a \qquad \text{for each} \qquad g \in G$$

holds only for the multiples of unit (for $a = 0$ if A has not a unit element).

For any C^*-dynamical system (A, G, σ) we can introduce the following $*$-algebras:

$$A_0^\infty = \{a \in A_0 : \text{the mapping } G \ni g \longmapsto \sigma_g a \in A \text{ is of the class } C^\infty\}$$

$$A^\infty = \{a \in A : a A_0^\infty \subset A_0^\infty, A_0^\infty a \subset A_0^\infty\} \tag{2}$$

$$M(A)^\infty = \{a \in M(A) : a A^\infty \subset A^\infty, A^\infty a \subset A^\infty\}. \tag{3}$$

If $A = C_\infty(\Lambda)$ and σ is transitive, then A_0^∞, A^∞ and $M(A)^\infty$ are composed of smooth (with respect to the induced differential structure) elements of A_0, A and $M(A)$, respectively. In a general case we consider A^∞ (the set of «smooth elements» of A) as the analogue of a differential structure (induced by the action of G on the pseudospace A). We call the pair (A, A^∞) shortly a *pseudomanifold*. Pseudomanifolds with morphisms defined by

$$\operatorname{Mor}((A, A^\infty), (B, B^\infty)) = \{\varphi \in \operatorname{Mor}(A, B) : \tilde{\varphi}(M(A)^\infty) \subset M(B)^\infty\}$$

form a category.

We want to emphasize the role of A_0 in this construction. In [2] one defines «smooth elements of A» without using A_0, but this leads to the structure which is not equivalent to the original differential structure in the commutative case, i.e. when $A = C_\infty(\Lambda)$ and Λ is a differential manifold.

Several statements which are true in the commutative case are expected to remain valid in a general case. The validity of some of those statements would be much desired, since it would simplify the whole structure. In the sequel such statements will be called *conditions of consistency*. At present, we do not know what should be assumed about (A, G, σ) in order to satisfy those conditions.

For instance, as a simple consequence of definitions, we have for any (A, G, σ) the following inclusions

$$A_0^\infty \subset A_0 \cap A^\infty$$

$$\{a \in M(A) : aA_0^\infty \subset A_0^\infty, A_0^\infty a \subset A_0^\infty\} \subset M(A)^\infty \tag{4}$$

$$M(A)^\infty \subset \{a \in M(A) : a(A_0 \cap A^\infty) \subset A_0 \cap A^\infty, (A_0 \cap A^\infty)a \subset A_0 \cap A^\infty\} \tag{4'}$$

(in order to prove the last inclusion we notice that A_0 is an ideal of $M(A)$, because A_0 is an algebraic ideal of A [6]). Now we consider the equality

$$A_0^\infty = A_0 \cap A^\infty \tag{5}$$

as a condition of consistency. If it is satisfied, we can say that A_0^∞ is composed of «smooth elements of A with compact supports». This condition also implies the equalities in (4) and (4′), so that $M(A)^\infty$ can be equivalently defined by replacing A^∞ in (3) by A_0^∞. In a similar manner we treat the following condition:

$$A^\infty = A \cap M(A)^\infty \tag{6}$$

(trivially $A^\infty \subset A \cap M(A)^\infty$). Next example of a consistency condition is the density of A^∞ in A:

$$\overline{A^\infty} = A. \tag{7}$$

Another consistency problem arises when we ask what is the analogue of a bounded real smooth function. First of all we accept the following analogy

$$\mathscr{D}(A, A^\infty) = \mathrm{Mor}\,((C_\infty(\mathbb{R}), C_\infty^\infty(\mathbb{R})), (A, A^\infty)) - \text{«real smooth functions»},$$

where $C_\infty^\infty(\mathbb{R}) = C_\infty(\mathbb{R}) \cap C^\infty(\mathbb{R})$ is the set of complex smooth functions on $\mathbb{R}$, vanishing at infinity. Since for any $a = a^* \in M(A)$ there exists a smooth function $f \in C_0^\infty(\mathbb{R})$ such that $f(a) = a$, we have the following inclusions

$$\mathscr{D}(A, A^\infty) \cap \mathrm{Re}\, M(A) \subset \mathrm{Re}\, M(A)^\infty$$

$$\mathscr{D}(A, A^\infty) \cap \mathrm{Re}\, A \subset \mathrm{Re}\, A^\infty \tag{8}$$

$$\mathscr{D}(A, A^\infty) \cap \mathrm{Re}\, A_0 \subset \mathrm{Re}\, A_0^\infty \tag{9}$$

(to prove (8) and (9) we use (6) and (5), respectively). Now we consider the equality

$$\mathscr{D}(A, A^\infty) \cap \mathrm{Re}\, M(A) = \mathrm{Re}\, M(A)^\infty \tag{10}$$

as a condition of consistency. If it is satisfied, the three above inclusions become equalities, and we can establish the analogy between $\operatorname{Re} M(A)^\infty$, $\operatorname{Re} A^\infty$, $\operatorname{Re} A_0^\infty$ and «smooth real functions» which are «bounded», «vanishing at infinity» and «with compact supports», resp. Let us notice that (10) is equivalent to the following condition:

(11) $\qquad$ if $a = a^* \in M(A)^\infty$ and $f \in C^\infty(\mathbb{R})$ then $f(a) \in M(A)^\infty$.

Consistency conditions (7), (5) together with (2), (6) and (11) can be taken as a starting point in an axiomatic approach to pseudomanifolds.

3. QUANTUM MECHANICS

In this section we show how a C^*-dynamical system arises naturally in quantum mechanics and we study the induced differential structure.

We consider a physical system whose classical *configuration space* is a finite-dimensional affine space Q. Then the corresponding classical *phase space* is the symplectic manifold $\mathscr{P} = T^*Q = Q \times V^*$ (with the canonical symplectic 2-form ω), where V is the tangent vector space of Q and V^* is the dual space of V.

The quantum description of our system is provided by an irreducible representation of the canonical commutation relations (CCR). We remind below the adequate definition of a representation of the CCR. To this end let us denote by $\mathscr{F}(\mathscr{P})$ the space of affine functions on $\mathscr{P}$ («affine» means «linear plus constant»). This space has a natural decomposition $\mathscr{F}(\mathscr{P}) = \mathscr{F}(Q) \oplus V$, defined as follows:

$$F = f \oplus v \ \text{ iff } \ F(q,p) = f(q) + \langle p, v\rangle \ \text{ for each } \ (q,p) \in Q \times V^*,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between V^* and V. The Poisson bracket of two affine functions $F_1 = f_1 \oplus v_1$ and $F_2 = f_2 \oplus v_2$ is equal to the following constant:

$$\{F_1, F_2\} = \langle \mathrm{d}f_1, v_2\rangle - \langle \mathrm{d}f_2, v_1\rangle.$$

A pair (W, H) is called *a representation of the CCR for* $\mathscr{P}$ if H is a separable Hilbert space and W is a strongly continuous mapping $\mathscr{F}(\mathscr{P}) \ni F \longmapsto W(F) \in \in \operatorname{Aut} H$, such that

$$W(F_1 + F_2) = W(F_1)\, W(F_2) \exp\left(\frac{i}{2}\{F_1, F_2\}\right)$$

and

$$W(\lambda \cdot 1) = e^{-i\lambda} I \ \text{ for real } \lambda,$$

where I is the identity of H.

It is well known that there exists essentially only one irreducible representation of the CCR for $\mathscr{P}$. More precisely, if (W_1, H_1) and (W_2, H_2) are two irreducible representations of the CCR for $\mathscr{P}$, then there exists a unitary operator $U : H_1 \to H_2$, unique up to a «phase factor» (complex number of modulus 1), such that $W_2(F) = U W_1(F) U^*$ for $F \in \mathscr{F}(\mathscr{P})$.

One can use for instance the Schrödinger's «position representation». In this case H is the space of square-integrable half-densities on Q and $W(F)$ for $F = f \oplus v$ is given by the following formula:

$$(W(F)\psi)(q) = e^{\frac{i}{2}\langle \mathrm{d}f, v\rangle - if(q)}\, \psi(q - v) \quad \text{for} \quad \psi \in H.$$

Now we pass to the description of the «non-commutative geometry» associated with this quantum-mechanical framework. First of all, the classical locally compact space $\mathscr{P}$ must be replaced by a pseudospace. The adequate choice is the C^*-algebra A composed of all compact operators on H (operators that «vanish at infinity»). By the equivalence theorem, A does not depend on particular irreducible representation (W, H) of the CCR for $\mathscr{P}$. One can show that

$M(A)$ is composed of all bounded operators on H,

Mor $(C_\infty(\mathbb{R}), A)$ can be identified with the set of all (not necessarily bounded) self-adjoint operators on H,

A_0 is composed of all operators of finite rank in H (see [5]).

We describe now a natural action of the additive Lie group $G = V \oplus V^*$ on A. Let us note first that G acts transitively on $\mathscr{P}$ by translations. For each $X \in G$ and any function h on $\mathscr{P}$ we denote by $\tau_X h$ the translation of h by X. Then for a fixed $X \in G$, the pair $(W \circ \tau_X, H)$ is again an irreducible representation of the CCR for $\mathscr{P}$. Hence there exists $U \in \mathrm{Aut}\, H$, unique up to a phase factor, such that $W \circ \tau_X = U W(\,\cdot\,) U^*$. The formula

(12) $$\sigma_X a = U a U^*$$

defines now the *translation of an operator* $a \in A$ by X. Using the commutation relations it is easy to check that $U = W(F)$, where $F \in \mathscr{F}(\mathscr{P})$ is such that

(13) $$X \lrcorner\, \omega = -\mathrm{d}F.$$

Formula (12) can be then written as follows:

$$\sigma_X a = W(F) a\, W(F)^*,$$

with F as in (13). From the strong continuity of W can be deduced the norm-

-continuity of $X \longmapsto \sigma_X a$ for each $a \in A$, so that (A, G, σ) is a C^*-dynamical system. By the irreducibility of W, the action of G on A is ergodic.

PROPOSITION 1. *For (A, G, σ) described above*

$$A_0^\infty = \{a \in A_0 : a H^\infty \subset H^\infty, a^* H^\infty \subset H^\infty\}$$

where $H^\infty = \{\psi \in H :$ *the mapping* $\mathcal{F}(\mathcal{P}) \ni F \mapsto W(F)\psi \in H$ *is of the class* $C^\infty\}$. ■

Proof. (i) If $a \in A_0^\infty$ and $\psi \in H^\infty$ then the mapping $\mathcal{F}(\mathcal{P}) \ni F \longmapsto W(F)(a\psi) = W(F) a W(F)^* \cdot W(F)\psi \in H$ is of the class C^∞.

(ii) If $a \in A_0$, $a H^\infty \subset H^\infty$ and $a^* H^\infty \subset H$ then a is a finite sum of operators of the form $|\psi_1\rangle\langle\psi_2|$ (Dirac's notation), where $\psi_1, \psi_2 \in H^\infty$. Such operators belong to A_0^∞. ■

PROPOSITION 2.

$$A^\infty = \{a \in A, : a H^\infty \subset H^\infty, a^* H^\infty \subset H^\infty\}$$

and

$$M(A)^\infty = \{a \in M(A) : a H^\infty \subset H^\infty, a^* H^\infty \subset H^\infty\}.$$ ■

Proof. If $a \in A^\infty$ and $\psi \in H^\infty$ then $a|\psi\rangle\langle\psi| \in A_0^\infty$, so

$$a|\psi\rangle\langle\psi|\psi = \|\psi\|^2 a\psi \in H^\infty.$$

If $a \in A$, $a H^\infty \subset H^\infty$ and $a^* H^\infty \subset H^\infty$, then for each $b \in A_0^\infty$

$$(ab)H^\infty \subset aH^\infty \subset H^\infty, \qquad (ab)^* H^\infty \subset b^* H^\infty \subset H^\infty,$$

$$(ba)H^\infty \subset H^\infty \quad \text{and} \quad (ba)^* H^\infty \subset a^* H^\infty \subset H^\infty.$$

The case of $M(A)^\infty$ can be treated identically. ■

These results show that the whole information about (A, A^∞) is contained in the pair (H, H^∞) and vice versa. Let us study this connection in more detail.

Let Q_1 and Q_2 be two configuration spaces, as Q was above. All derived objects like A, A^∞ etc. will be also labelled by the corresponding number (1 or 2). We introduce the following notation:

Iso (H_1, H_2) – the set of unitary operators from H_1 to H_2
Iso $((H_1, H_1^\infty), (H_2, H_2^\infty)) = \{U \in \text{Iso}\,(H_1, H_2) : UH_1^\infty = H_2^\infty\}$
Iso (A_1, A_2) – isomorphisms in the category of pseudospaces.

PROPOSITION 3. *There is a bijective (modulo phase factor) correspondence between $\varphi \in \text{Iso}\,(A_1, A_2)$ and $U \in \text{Iso}\,(H_1, H_2)$ given by $\varphi(a) = U a U^*$ for $a \in A_1$. If φ and U are related in this way, then*

$$\varphi \in \text{Iso}\,((A_1, A_1^\infty), (A_2, A_2^\infty)) \textit{ iff } U \in \text{Iso}\,((H_1, H_1^\infty), (H_2, H_2^\infty)). \qquad \blacksquare$$

Proof. All irreducible representations of A_1 (the C^*-algebra of compact operators in H_1) are unitarily equivalent to the identical representation (c.f. [7]). This is also the case for the representation of A_1 in H_2 given by $\varphi \in \text{Iso}\,(A_1, A_2)$. This proves the existence of U. If $UH_1^\infty = H_2^\infty$ then

$$\tilde{\varphi}(M(A_1)^\infty) = M(A_2)^\infty$$

by Proposition 2. Conversely, if the above condition is satisfied, then for $\psi \in H_1^\infty$

$$|U\psi\rangle\langle U\psi| = U(|\psi\rangle\langle\psi|)U^* = \varphi(|\psi\rangle\langle\psi|) \in M(A_2)^\infty,$$

hence $U\psi \in H_2^\infty$, by Proposition 2. ■

It is interesting to consider H^∞ as a topological vector space. The natural locally convex topology on H is induced by the family of seminorms given by the norms of partial derivatives of $F \longmapsto W(F)\psi$ at $F = 0$. It is easy to see, that in the Schrödinger representation H^∞ coincides with the Schwartz space $\mathscr{S}(Q)$ of functions on Q (multiplied by a constant half-density on Q). Also the above described topology on H^∞ coincides with the standard topology of $\mathscr{S}(Q)$. The question now arises whether this topology is preserved by an isomorphism or not. The following proposition shows that the answer is affirmative.

PROPOSITION 4. *If $U \in \text{Iso}\,(H_1, H_2)$ and $UH_1^\infty = H_2^\infty$ then $U|_{H_1^\infty}$ is a homeomorphism of H_1^∞ and H_2^∞.* ■

Proof. Since H_1^∞ and H_2^∞ are Fréchet spaces, one can use The Closed Graph Theorem. It suffices to show that the graph of $U|_{H_1^\infty}$ is closed in $H_1^\infty \oplus H_2^\infty$. Let $\psi_n \to 0$ in H_1^∞ and $U\psi_n \to \chi$ in H_2^∞, then for any $h \in H_2$ the scalar product

$$\langle \chi | h \rangle = \lim_{n\to\infty} \langle U\psi_n | h \rangle = \lim_{n\to\infty} \langle \psi_n | U^* h \rangle = 0,$$

hence $\chi = 0$. ■

This result shows that the topology of $H^\infty = \mathscr{S}(Q)$ is already incorporated in the structure of the pseudomanifold (A, A^∞). We conclude that the equivalent description of (A, A^∞) is provided by the following «Gelfand triple» (or «rigged Hilbert space»):

$$H^\infty \subset H \subset (H^\infty)^x$$

(here $(H^\infty)^x$ denotes the space of continuous antilinear functionals on H^∞).

PROPOSITION 5. *$M(A)^\infty$ is composed of such bounded operators a in H that $a|_{H^\infty}$ and $a^*|_{H^\infty}$ map continuously H^∞ into H^∞.*

Proof. The argument is the same as in the proof of Proposition 4. ■

Let us now consider the consistency conditions of the preceding section. It is easy to prove that our pseudomanifold (A, A^∞) satisfies conditions (5), (6) and (7). To this end we use Prop. 1, Prop. 2 and the density of H^∞ in H. At present, the author is not able to prove condition (11). It is clear that this would be a quite interesting theorem. For example, it would imply that a one-parameter group oi unitary transformations preserves $\mathscr{S}$ if its generator does. Let us note here some partial results concerning this problem. The first one is the trivial observation that for our (A, A^∞) we have the equality in (9). The second one is the following

PROPOSITION 6. *If $a = a^* \in M(A)^\infty$ is diagonal*

(i) *in the «position representation»*

or

(ii) *in the basis composed of the «Hermite functions»,*

then for any $f \in C^\infty(\mathbb{R})$ we have $f(a) \in M(A)^\infty$. ■

Proof. (i) If a is an operator of multiplication by a function $a(q)$, then one can show that this function and all its derivatives (a must be smooth) are polynomially bounded. Then $f \circ a$ has similar properties.

(ii) We have

$$a = \sum_n \lambda_n |h_n\rangle\langle h_n|,$$

where λ_n is a bounded sequence and h_n are the normalized Hermite functions. Then

$$f(a) = \sum_n f(\lambda_n) \, | h_n \rangle \langle h_n |$$

and $f(\lambda_n)$ is a bounded sequence. For any $\psi \in H$, ψ belongs to H^∞ if and only if $\langle h_n | \psi \rangle$ is a rapidly decreasing sequence ([8]). It follows that for $\psi \in H^\infty$, $f(a)\psi = \sum_n f(\lambda_n) \langle h_n | \psi \rangle h_n$ belongs to H^∞. ■

Finally, let us remark that all pseudomanifolds (A, A^∞) considered in this section are mutually isomorphic. This is so because there exist unitary operators belonging to Iso $((H_1, H_1^\infty), (H_2, H_2^\infty))$ for all Q_1 and Q_2 (see [8]). In particular, the pair (A, A^∞) does not «remember» the dimension of Q.

REFERENCES

[1] S.L. WORONOWICZ, *Pseudospaces, Pseudogroups and Pontriagin duality*, Lecture Notes in Physics, **116**, Berlin 1980.

[2] A. CONNES, *C*-Algèbres et Géometrié Différentielle*, IHES preprint, 1980.

[3] S. ZAKRZEWSKI, *Unitary Relations*, Publicacoes de Fisica Matematica, **4**, Coimbra 1985.

[4] W.M. TULCZYJEW, S. ZAKRZEWSKI, *The category of Fresnel kernels*, Journal of Geometry and Physics, vol. 1, 3.

[5] G.K. PEDERSEN, *Measure theory for C*-algebras*, Math. Scand. **19**, (1966), 131 - 145.

[6] G.K. PEDERSEN, N.H. PETERSEN, *Ideals in a C*-algebra*, Math. Scand. 27 (1970), 193 - 204.

[7] J. DIXMIER, *Les C*-algèbres et leur représentations*, Paris 1969.

[8] B. SIMON, *Distributions and Their Hermite Expansions*, J. Math. Phys. **12** (1971).

[9] G.K. PEDERSEN, *C*-algebras and their automorphism groups*, Acad. Press, London-New York-San Francisco, 1979, §3.

Manuscript received: July 15, 1985.